SOME IDENTITIES OF SYMMETRY FOR q-EULER POLYNOMIALS UNDER THE SYMMETRIC GROUP OF DEGREE n ARISING FROM FERMIONIC p-ADIC q-INTEGRALS ON $\mathbb{Z}_p$

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ABSTRACT. In this paper, we investigate some new symmetric identities for the q-Euler polynomials under the symmetric group of degree n which are derived from fermionic p-adic q-integrals on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed prime number such that $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$ and the q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

As is well known, the Euler numbers are defined by

$$E_0 = 1, \quad (E+1)^n + E_n = 2\delta_{0,n}, \quad (n \in \mathbb{N} \cup \{0\}),$$

with the usual convention about replacing E^n by E_n(see [1–14]).

The Euler polynomials are given by

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l = (E+x)^n, \quad (n \geq 0), \quad (\text{see } [1, 2]).$$

In [8], Kim introduced Carlitz-type q-Euler numbers as follows:

$$\mathcal{E}_{0,q} = 1, \quad q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = [2]_q \delta_{0,n}, \quad (n \geq 0), \quad (\text{see } [8]), \tag{1.1}$$

with the usual convention about replacing $\mathcal{E}_q^n$ by $\mathcal{E}_{n,q}$.

The Carlitz-type q-Euler polynomials are also defined as

$$\mathcal{E}_{n,q}(x) = \left(q^x \mathcal{E}_q + [x]_q\right)^n = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{E}_{l,q} [x]_q^{n-l}, \quad (\text{see } [3, 8]). \tag{1.2}$$

Let $C(\mathbb{Z}_p)$ be the space of all $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. Then, for $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) \tag{1.3}$$

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$$= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x$$

$$= \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (\text{see [5–11]}).$$

From (1.3), we note that

(1.4)

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (n\in\mathbb{N}), \quad (\text{see [8]}).$$

The Carlitz-type q-Euler polynomials can be represented by the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_{-q}(y), \quad (n\geq 0), \quad (\text{see [8]}). \tag{1.5}$$

Thus, by (1.5), we get

$$\begin{aligned} &\mathcal{E}_{n,q}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l \, d\mu_q(y) [x]_q^{n-l} \\ &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \mathcal{E}_{l,q} [x]_q^{n-l}, \quad (\text{see [8]}). \end{aligned} \tag{1.6}$$

From (1.4), we can easily derive

$$q\int_{\mathbb{Z}_p} [x+1]_q^n \, d\mu_{-q}(x) + \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_{-q}(x) = [2]_q \, \delta_{0,n}, \quad (n\in\mathbb{N}\cup\{0\}). \tag{1.7}$$

The equation (1.7) is equivalent to

$$q\mathcal{E}_{n.q}(1) + \mathcal{E}_{n,q} = [2]_q \, \delta_{0,n}, \quad (n\geq 0). \tag{1.8}$$

The purpose of this paper is to give some new symmetric identities for the Carlitz-type q-Euler polynomials under the symmetric group of degree n which are derived from fermionic p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetric identities for $\mathcal{E}_{n,q}(x)$ under S_n

Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv w_2 \equiv w_3 \equiv \cdots \equiv w_n \equiv 1 \pmod 2$. Then, we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1\cdots w_{n-1}}}(y) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{q^{w_1\cdots w_{n-1}}}} \\ &\quad \times \sum_{y=0}^{p^N-1} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} (-q^{w_1\cdots w_{n-1}})^y \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
=&\frac{1}{2}\lim_{N\to\infty}[2]_{q^{w_1\cdots w_{n-1}}}\\
&\times\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1}e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)(m+w_ny)+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n}\left(\prod_{\substack{i=1\\i\neq j}}^{n}w_i\right)k_j\right]_q t}\\
&\times(-1)^{m+y}q^{w_1\cdots w_{n-1}(m+w_ny)}.
\end{aligned}
$$

Thus, by (2.1), we get

$$
\begin{aligned}
&\frac{1}{[2]_{q^{w_1\cdots w_{n-1}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q t}d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&=\frac{1}{2}\lim_{N\to\infty}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1}(-1)^{\sum_{i=1}^{n-1}k_i+m+y}\\
&\times q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j+\left(\prod_{j=1}^{n-1}w_j\right)m+\left(\prod_{j=1}^{n}w_j\right)y}\\
&\times e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)(m+w_ny)+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j\right]_q t}.
\end{aligned}\tag{2.2}
$$

As this expression is invariant under any permutation $\sigma\in S_n$, we have the following theorem.

Theorem 2.1. *Let $w_1,w_2,\ldots,w_n\in\mathbb{N}$ such that $w_1\equiv w_2\equiv\cdots\equiv w_n\equiv 1 \pmod 2$. Then, the following expressions*

$$
\begin{aligned}
&\frac{1}{[2]_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j}\\
&\times\int_{\mathbb{Z}_p}e^{\left[\left(\prod_{j=1}^{n-1}w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_{\sigma(i)}\right)k_j\right]_q t}d\mu_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}(y)
\end{aligned}
$$

are the same for any $\sigma\in S_n$, $(n\geq 1)$.

Now, we observe that

$$
\begin{aligned}
&\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_j\right)k_j\right]_q t\\
&=\left[\prod_{j=1}^{n-1}w_j\right]_q\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}.
\end{aligned}\tag{2.3}
$$

By (2.3), we get

$$\begin{aligned}(2.4)\quad &\int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j)y+(\prod_{j=1}^{n} w_j)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&=\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1} w_j\right]_q^m \int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}^m d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1} w_j\right]_q^m \mathcal{E}_{m,q^{w_1\cdots w_{n-1}}}\left(w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\frac{t^m}{m!}.\end{aligned}$$

For $m\geq 0$, from (2.4), we have

$$\begin{aligned}(2.5)\quad &\int_{\mathbb{Z}_p}\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q^m d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
&=\left[\prod_{j=1}^{n-1} w_j\right]_q^m \mathcal{E}_{m,q^{w_1\cdots w_{n-1}}}\left(w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right),\quad (n\in\mathbb{N}).\end{aligned}$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following theorem.

Theorem 2.2. *Let $w_1,\dots w_n\in\mathbb{N}$ be such that $w_1\equiv w_2\equiv\cdots\equiv w_n\equiv 1 \pmod 2$. For $m\geq 0$, the following expressions*

$$\begin{aligned}&\frac{\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^m}{[2]_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}q^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j}\\
&\times\mathcal{E}_{m,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x+w_{\sigma(n)}\sum_{j=1}^{m-1}\frac{k_j}{w_{\sigma(j)}}\right)\end{aligned}$$

are the same for any $\sigma\in S_n$.

It is not difficult to show that

(2.6)

$$\begin{aligned}&\left[y+w_nx+w_n\sum_{j=0}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}\\
=&\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_{q^{w_n}}+q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}.\end{aligned}$$

Thus, by (2.6), we get

(2.7)

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=0}^{n-1}\frac{k_j}{w_j}\right]^m_{q^{w_1\cdots w_{n-1}}} d\mu_{q^{-w_1\cdots w_{n-1}}}(y)\\
=&\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}} q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}[y+w_nx]^l_{q^{w_1\cdots w_{n-1}}}\,d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
=&\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}}\\
&\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j}\mathcal{E}_{l,q^{w_1\cdots w_{n-1}}}(w_nx).
\end{aligned}$$

From (2.7), we have

(2.8)

$$\begin{aligned}
&\frac{\left[\prod_{j=1}^{n-1}w_j\right]^m_q}{[2]_{q^{w_1\cdots w_{n-1}}}}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{l=1}^{n-1}k_l}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]^n_{q^{w_1\cdots w_{n-1}}} d\mu_{-q^{w_1\cdots w_{n-1}}}(y)\\
=&\sum_{l=0}^{m}\binom{m}{l}\frac{\left[\prod_{j=1}^{n-1}w_j\right]^l_q}{[2]_{q^{w_1\cdots w_{n-1}}}}[w_n]_q^{m-l}\,\mathcal{E}_{l,q^{w_1\cdots w_{n-1}}}(w_nx)\\
&\times\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}(-1)^{\sum_{j=1}^{n-1}k_j}q^{(l+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}}\\
=&\frac{1}{[2]_{q^{w_1w_2\cdots w_{n-1}}}}\sum_{l=0}^{m}\binom{m}{l}\left[\prod_{j=1}^{n-1}w_j\right]^l_q[w_n]_q^{m-l}\,\mathcal{E}_{l,q^{w_1\cdots w_{n-1}}}(w_nx)\\
&\times\hat{T}_{m,q^{w_n}}(w_1,w_2,\ldots,w_{n-1}\mid l),
\end{aligned}$$

where

$$\hat{T}_{m,q}(w_1,\ldots,w_{n-1}\mid l) \tag{2.9}$$

$$= \prod_{s=1}^{n-1} \sum_{k_s=0}^{w_s-1} q^{(l+1)\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} \left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q^{m-l} (-1)^{\sum_{j=1}^{n-1} k_j}.$$

As this expression is invariant under any permutation in S_n, we have the following theorem.

Theorem 2.3. *Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ be such that $w_1 \equiv w_2 \equiv \cdots \equiv w_n \equiv 1 \pmod 2$. For $m \geq 0$, the following expressions*

$$\frac{1}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}} \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^{l} \left[w_{\sigma(n)}\right]_q^{m-l}$$
$$\times \mathcal{E}_{l,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x\right) \hat{T}_{m,q^{w_{\sigma(n)}}}\left(w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n-1)} \mid l\right)$$

are the same for any $\sigma \in S_n$.

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